Disc-counting enumerative invariants: ①

For spheres (or closed Riem. surfaces), $\overline{\mathcal{M}}_{g,k}(M,J,\beta)$ compact, pseudocycle (i.e. bubbling $\mathrm{codim}_{\mathbb{R}} 2$)

$\Rightarrow$ get actual invariants by $\int$ cohomology classes over moduli spaces.

For discs w/ Lagr. boundary, this is no longer true. Expect codim. 1 boundary = disc bubbling:

$$\partial(\overline{\mathcal{M}}(M,L,J,\beta)) = \{ \bigcirc\!\!\bigcirc_L \} = \bigsqcup_{\beta=\beta_1+\beta_2} \overline{\mathcal{M}}_{0,1}(L,\beta_1) \underset{ev}{\times} \overline{\mathcal{M}}_{0,1}(L,\beta_2)$$

$\dim_{\mathbb{R}} d = n-3+\mu(\beta)$

(only holds if regularity holds, or after perturbation...)

$\dim_{\mathbb{R}} \; (n-2+\mu(\beta_1)) + (n-2+\mu(\beta_2)) - n = d-1.$

matching constraint at node

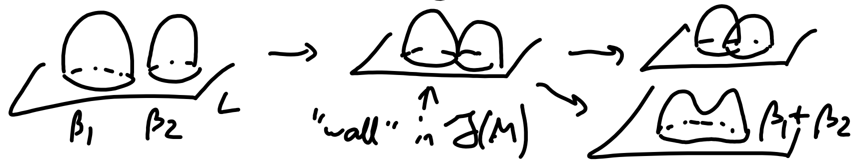
One could try to define "open GW" disc counts,

eg. in Calabi-Yau 3-fold: M^6, $c_1(TM)=0$, if L has vanishing Maslov class (eg. SLag: $\arg(\Omega_{|L}) =$ constant, or more generally, $\arg(\Omega_{|L}) : L \to S^1$ lifts to $\mathbb{R}$-valued) $\Rightarrow \mu(\beta)=0 \ \forall\beta$

so expected dim $\mathcal{M}(L,\beta) = n-3+\mu(\beta) = 0$. Could hope to define $N_\beta = \#\mathcal{M}(L,\beta)$?

- If β is divisible, $\beta = k\beta_1$, there is an issue with regularity of multiple covers (expect k-fold covers contribution to N_β should be $\frac{1}{k^2}\dots$) $\to N_\beta \in \mathbb{Q}$ at best.
- More importantly, the space of J's for which boundaries of simple J-hol. discs intersect is $\mathrm{codim}_{\mathbb{R}} 1$ in $\mathcal{J}$.

so there are transitions



$N_{\beta_1+\beta_2}$ jumps!

N_β is guaranteed to be an invariant only when $[\omega]\cdot\beta = \min([\omega]\cdot\pi_2(M,L) \cap \mathbb{R}_+)$, or index argument to exclude bubbling.

- Similarly for more general open GW invts with incidence conditions in $H^*(M)$ (interior) or $H^*(L)$ (boundary) ie. $\mathcal{M}_{\ell,k}(L,J,\beta) \xrightarrow{ev} M^\ell \times L^k$, $\int_{\mathcal{M}(\beta)} \prod ev^*\alpha_i \wedge \prod evb^*\gamma_i$ - issue is $[\mathcal{M}(\beta)]$ isn't a cycle !!!

$\to$ Defining disc-counting invariants in this way requires extra data - done only in certain cases (S^1-actions: Katz-Liu, antihol. involutions: Welschinger, Georgieva-Zinger, ...)

general formalism = "bounding cochains" (FOOO) (see eg. Solomon-Tukachinsky for OGW this way).

= algebraic formalism to complete $\mathcal{M}(L,\beta)$ to a cycle that one can $\int$ on. (requires extra data! not always $\exists$)

★ One case where things are elementary: assume $L \subset (M,\omega)$ is <u>monotone</u>, oriented

ie. $\exists \lambda > 0$ st. $\forall \beta \in \pi_2(M,L)$ disc class, $\omega\cdot\beta = \lambda\mu(\beta)$. (<u>NB</u>: this forces M monotone, ie. $\omega = \frac{\lambda}{2}c_1$ on $\pi_2(M)$).

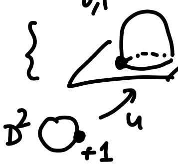
<u>Cor</u>:
- no discs of Maslov index ≤ 0
- no disc bubbling when considering $\mathcal{M}(L,\beta)$ for $\mu(\beta)=2$!
  (in general: this happens for <u>lowest area</u> holom. curves and/or lowest index curves! if min index >0)
- no multiple covers (recall μ even for L oriented)
  $\Rightarrow$ Maslov index 2 discs are regular for generic J.

⚠ still have possible sphere bubbling if $\partial\beta = 0$, could have discs $\to$ constant disc + $c_1 = 1$ sphere.

But this is $\mathrm{codim}_{\mathbb{R}} 2$ under evaluation map $\Rightarrow$ not an issue.

This issue aside, for regular J and for $\mu(\beta)=2$, $\mathcal{M}_{0,1}(L,J,\beta)$ is a compact closed mfld of dim. n.

evaluation map: $ev : \mathcal{M}_{0,1}(L,J,\beta) \longrightarrow L$, $u \mapsto u(1)$

$\left\{ \right.$  $\left. \right\} / Aut(D^2,1)$ $\quad$ map between two closed n-manifolds.

If we have an orientation on $\mathcal{M}_{0,1}(L,\beta)$, then can define $n_\beta(L) := \deg(ev) \in \mathbb{Z}$
($= \#$ J-hol. discs in class β whose boundary passes through a generic point of L).
(in absence of orientation, only $\mathbb{Z}/2$).

⚠ Determining an orientation on $\mathcal{M}_{0,k}(L,\beta)$ requires the choice of a <u>spin structure</u> on L

↳ orientation on $X \equiv$ homotopy class of (stable) trivialization of TX over 0-skeleton which extends over the 1-skeleton.

(ie: $T_pX \xrightarrow{\sim} \mathbb{R}^n$ homotopy class of iso $\leftrightarrow$ choice of orientation of T_pX
orientation of $X \leftrightarrow$ choose these consistently at points in same conn. component).

obstruction: $w_1(TX) \in H^1(X,\mathbb{Z}/2)$. set of choices: $H^0(X,\mathbb{Z}/2)$.

spin structure $\equiv$ homotopy class of (stable) triv.^{n} of TX over 1-skeleton which extends over the 2-skeleton.

ie. $\forall$ loop $\gamma : S^1 \to X$, pick a triv.^{n} $\gamma^* TX \oplus \underline{\mathbb{R}}^k \xrightarrow{\sim} \underline{\mathbb{R}}^{n+k}$, consistently.
can ignore if $n \geq 3$.
(issue is for $n=2$, $\pi_1 SO(2) = \mathbb{Z}$ instead of $\mathbb{Z}/2$)

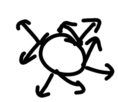
obstruction: $w_2(TX) \in H^2(X,\mathbb{Z}/2)$; set of choices: $H^1(X,\mathbb{Z}/2)$.

(NB: $w_2 = 0$ for all smooth oriented mflds of dim ≤ 3).

This is equivalent to an orientation of the loop space $\mathcal{L}X$.

A choice of spin structure leads to an orientation on the index line ($\Lambda^{top}\ker \otimes \Lambda^{top}\mathrm{coker}^*$) of the linearized operator D_u on $W^{k,p}(D^2,\partial D^2; u^*TM, u^*TL)$, by determining a preferred choice of trivialization of u^*TL at the boundary. (see Seidel, FOOO, ...).

★ The 2 spin structures on S^1:

1) $TS^1 \cong \underline{\mathbb{R}} \Rightarrow TS^1 \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^2$  "standard"

2) $TS^1 \oplus \underset{NS^1}{\underline{\mathbb{R}}} \cong \underline{\underline{\mathbb{R}}}^2$  "bounding"

Sign change formula: if 2 spin structures differ by $\sigma \in H^1(L,\mathbb{Z}/2)$, orientation on $\mathcal{M}(L,\beta)$ changes by $(-1)^{\langle\sigma,\partial\beta\rangle}$ & hence so does $n_\beta(L)$.

Concretely, β $S^1 \subset \mathbb{R}^2$ $\quad n_\beta = +1$ for standard spin str.
$\qquad\qquad -1$ bounding

★ The <u>superpotential</u> for a monotone (oriented, spin) Lagrangian $L \subset (M,\omega)$:

$$W_L = \sum_{\beta \,/\, \mu(\beta)=2} n_\beta(L)\, z^{\partial\beta} \in \mathbb{Z}[H_1(L)]$$

ie. Laurent polynomial; if $\gamma_1 \dots \gamma_r$ basis of $H_1(L)$, $\partial\beta = \sum m_i \gamma_i \Rightarrow$ contr. $z_1^{m_1} \dots z_r^{m_r}$.

(This can be viewed as related to univ. Novikov weights q^β, but only keeps image of β under $H_2(M,L) \xrightarrow{\partial} H_1(L)$. In $\mathbb{R}^{2n}$ or $\mathbb{CP}^n$ no information lost given $\mu(\beta)=2$).

Prop: For monotone L, W_L is independent of J and invariant under monotone Lagrangian isotopies.

(Pf.: $\coprod_{t\in[0,1]} \mathcal{M}_{0,1}(L_t, J_t, \beta)$ gives a cobordism between $t=0$ & $t=1$, without any extra boundary components since no disc bubbling. Hence $n_\beta(L)$ invariant).

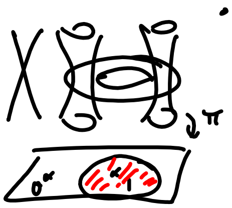
$\Rightarrow$ can use W_L to distinguish monotone Lagrangians.

Examples:

- $S^1 \subset \mathbb{R}^2$  $n_\beta = 1 \rightsquigarrow W = z$.
- $S^1 \subset \mathbb{CP}^1$  $n_{\beta_1} = n_{\beta_2} = 1 \rightsquigarrow W = z + z^{-1}$.

  monotone iff equal areas
- $S^1(r) \times \dots \times S^1(r) \subset (\mathbb{C}^n, \omega_0)$ $\quad u = (u_1, \dots, u_n) : D^2 \to (\mathbb{C}^n, L)$ each component is a holom disc in $\mathbb{R}^2$ w/ boundary in $S^1(r)$, $\mu(u) = \sum \mu(u_i) \Rightarrow$
  - monotone iff all r equal
  - $\mu=2$ discs are constant in all but one factor, which is .

  So: n classes $\beta_1 = [D^2 \times pt], \dots, \beta_n$ with $n_{\beta_i} = 1$ (one disc through each pt of L, all regular)

  $W = z_1 + \dots + z_n$
-  by contrast, Chekanov(-Schlenk) tori $L = \{|x_1| = \dots = |x_n|, \ |\prod x_i - 1| = r\} \subset \mathbb{C}^n$: $(r<1)$

  use projection $\pi : \mathbb{C}^n \to \mathbb{C}$, $(x_1, \dots, x_n) \mapsto \prod x_i$

  $\pi \circ u$ is a holom. disc in $\mathbb{C}$ with boundary on circle of radius r at 1.

  Can check $\mu(u) = \mu(\pi\circ u) = 2 \cdot \deg(\pi \circ u) = 2(\beta \cdot \Delta)$ where $\Delta = \{\prod x_i = 1\}$.

  Proof 1: T^{n-1} acts by rotations of x_i preserving $\prod x_i$, freely outside of coord. planes, the vf's of the S^1-actions give a trivialization of $\mathcal{F} := \ker d\pi \simeq t^{n-1} \otimes \mathbb{C}$ away from $\pi^{-1}(0)$. Consider short exact seq.

  $$0 \to u^*\mathcal{F} \to u^*TM \xrightarrow{d\pi} (\pi\circ u)^* T\mathbb{C} \to 0$$

  & subbundles $\quad u^*_{|\partial D^2} t^{n-1} \to u^*TL \xrightarrow{d\pi} (\pi \circ u)^* TS^1_r \to 0$

  additivity of Maslov index: $\quad 0 \qquad \mu(u) \qquad \mu(\pi\circ u)$

  Proof 2: the holom. vol. form $\Omega = \dfrac{dx_1 \wedge \dots \wedge dx_n}{\prod x_i - 1}$ gives a trivialization of $\det_{\mathbb{C}}(TM)$ away from Δ

  Wrt this trivialization, $\arg(\Omega_{|L}) = n\frac{\pi}{2}$ is constant $\Rightarrow (\det_{\mathbb{C}} TM, \det_{\mathbb{R}} TL) \cong (\underline{\mathbb{C}}, \underline{\mathbb{R}})$.

  This gives $\mu = 0$ if $u(D^2) \cap \Delta = \emptyset$. ($\Omega$ gives triv^n of $(\det u^*TM, \det u^*TL)$)

  otherwise each pole of Ω causes the triv^n of $\det u^*TM$ over D^2 to differ from that induced by Ω, hence $\det u^*TL$ no longer trivial subbundle of $\det u^*TM$: $\mu = 2 \cdot \#\text{poles} = 2[u]\cdot\Delta$.

Meanwhile: $\pi\circ u$ takes values in disc of radius r centered at 1 $\Rightarrow$ $x_1 \dots x_n$ have no zeroes along u. So $\frac{x_i}{x_j}\circ u : D^2 \to \mathbb{C}^*$ are nonvanishing holomorphic functions, with values in S^1 at the boundary (since $|x_i| = |x_j|$ on L). Open mapping principle $\Rightarrow \frac{x_i}{x_j} = \text{const} \in S^1$.

<u>So</u>: $x_j = e^{i\theta_j} (\pi\circ u)^{1/n}$.

constant   fix some choice of root over disc centered at 1.

($\bullet$ $\mu = 0 \Rightarrow \pi\circ u$ constant $\Rightarrow u$ constant. Already knew it, given L is monotone).

$\bullet$ $\mu = 2 \Rightarrow \pi\circ u(z) = 1 + rz$ up to reparametrization $\Rightarrow u(z) = \left(e^{i\theta_1}(1+rz)^{1/n}, \dots, e^{i\theta_n}(1+rz)^{1/n}\right)$

So there is a single class β_0 with $n_{\beta_0} = 1$ (check regularity & orientation), $W = z_0$.

This shows Chekanov(-Schlenk) tori are not Ham. isotopic to monotone product tori.

(<u>NB</u>: monotone because $\pi_2(\mathbb{C}^n, L) \simeq \mathbb{Z}^n$ generated by β_0 $\mu = 2$ $\omega(\beta_0) > 0$

and by $\underset{j=1\dots n-1}{\alpha_j} = \{(1-r)^{1/n}(1, \dots, 1, \underset{x_j}{z}, 1 \dots 1, \underset{x_n}{\bar{z}}), z \in D^2\}$ Lagrangian disc, $\mu = 0$, $\omega = 0$.)

(of course, products of S^1's and Chekanov tori give yet different examples with between 1 and n terms in W).

$*$ <u>in $\mathbb{CP}^n$</u>: monotone Clifford torus $\{(1 : x_1 : \dots : x_n) \mid |x_1| = \dots = |x_n| = 1\} = T^n_{Cl} \subset \mathbb{CP}^n$

Maslov index $= 2[u]\cdot\Delta$, Δ = union of coordinate hyperplanes

$\Rightarrow$ Maslov index 2 discs only intersect one coord. plane; all other $\frac{x_i}{x_j}$ are constant (hol. discs in $(\mathbb{C}^*, S^1)$)

so of the form $z \mapsto (e^{i\theta_0} : \dots : \underset{j\text{th}}{z} : \dots e^{i\theta_n})$ for some $j \in \{0..n\}$.

$(n+1)$ families of discs, $n(\beta_j) = 1$.

the boundaries of these discs represent classes $= (1,0,\dots,0), \dots, (0,\dots,0,1), (-1\dots-1)$ in $H_1(T^n) = \mathbb{Z}^n$

in fact $\sum_{j=0}^{n} \beta_j = [\mathbb{CP}^1] \in H_2(\mathbb{CP}^n, L) \simeq \mathbb{Z}^{n+1}$.

So: $W_{Cl} = z_1 + \dots + z_n + \frac{1}{z_1 \dots z_n}$.

suitably sized s.c.c.
not enclosing the origin

vs. Chekanov-Schlenk torus $\{(1 : x_1 : \dots : x_n) \mid |x_1| = \dots = |x_n|, \ \prod x_i \in \gamma\}$

$\Rightarrow$ recall from above in $\mathbb{C}^n$: $\pi_2(\mathbb{CP}^n, L)$ gen^d by β_0 $\mu = 2$, $\alpha_1 \dots \alpha_{n-1}$ $\mu = 0$, $[\mathbb{CP}^1]$ $\mu = 2(n+1)$. basis of $H_1(L)$ call variables $z, w_1 \dots w_{n-1}$

$$W_{Ch} = z + \frac{(1 + w_1 + \dots + w_{n-1})^n}{z^n w_1 \dots w_{n-1}}$$

the family of discs in $\mathbb{C}$

n^n discs through each pt, intersecting $\{x_0 = 0\}$ once & $\{x_1 \dots x_n = 0\}$ n times

e.g. in $\mathbb{CP}^2$, $W = z + \frac{(1+w)^2}{z^2 w} = z + \frac{1}{z^2 w} + \frac{2}{z^2} + \frac{w}{z^2}$ (4 homotopy classes, one of them has $n_\beta = 2$).

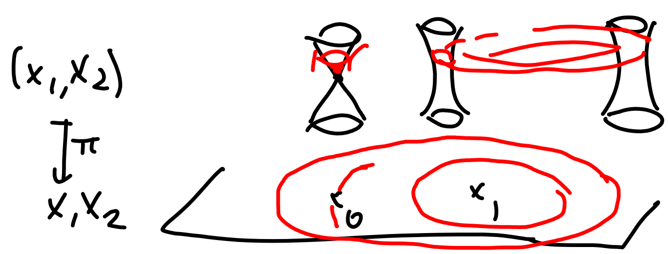
Vianna's $T_{1,4,25} \subset \mathbb{CP}^2$: $W = u + \frac{2(1+v)^2}{u^2} + \frac{(1+v)^5}{u^5 v}$ (10 homotopy classes, $\sum n_\beta = 41$).

These expressions are manifestly different... yet related by changes of variables!

$$W_{Cl} = z_1 + z_2 + \frac{1}{z_1 z_2} \xrightarrow[\substack{z = z_1+z_2 \\ w = z_2/z_1}]{} W_{Ch} = z + \frac{(1+w)^2}{z^2 w} \xrightarrow[\substack{u = z + \frac{1}{z^2 w} \\ v = \frac{1}{z^3 w}}]{\left(z = \frac{u}{1+v}\right)} W_{1,4,25} = u + \frac{2(1+v)^2}{u^2} + \frac{(1+v)^5}{u^5 v}.$$

The reason:
- these tori are Lagrangian isotopic (not Ham.! isotopy through non-monotone tori)
- as long as there are no discs of $\mu<2$, the counts n_β and hence W_L don't change, since no disc bubbling occurs.
- wall-crossing formula for W_L : jumps in $W_L \Leftrightarrow$ change of vars accounting for $\mu=0$ discs.

Consider again the Lagr. tori $T_{r,\lambda} = \{\pi(|x_1|^2 - |x_2|^2) = \lambda,\ |x_1 x_2 - 1| = r\} \subset \mathbb{C}^2$



By considering projection $\pi\circ u$ for $u: D^2 \to (\mathbb{C}^2, T_{r,\lambda})$,
- no discs with $\mu<0$
- discs with $\mu=0$ have $\pi\circ u = $ const.
- discs w/ $\mu=2$ have $\pi\circ u : D^2 \xrightarrow[\text{bihol.}]{} D(1,r)$

As long as $\nexists\ \mu=0$ discs, can follow $\mathcal{M}(L,\beta)$ for $\mu(\beta)=2$ as L moves in an isotopy, and $n_\beta = \deg(ev)$ doesn't jump. Jumps occur when $\mu=0$ discs can bubble off. Here this happens at $r=1$, when $T_{r,\lambda}$ passes through the fiber $x_1x_2=0$ and bounds a disc in the x_1 or x_2-plane (depending on sign of λ).

For $r<1$, Chekanov-type tori, only one class of discs w/ $n_{\beta_0}=1$, $W = z_0$ ("trivial" sections over $D(1,r)$)
For $r>1$, product-type tori, two families of discs, $W = z_1 + z_2$.
(when disc passes through fiber $x_1x_2=0$, one family β_1 passes through x_1-axis, the other β_2 through x_2-axis.

(in both cases, check by isotoping without crossing $x_1x_2=0$ to known monotone tori).

Consider case $\lambda>0$: as $r\downarrow$ through 1, $T_{r,\lambda}$ passes through x_1-plane (intersects in circle $\pi|x_1|^2=\lambda$)
→ The discs in class β_1 have lowest area (in fact $\omega(\beta_2)-\omega(\beta_1)=\lambda>0$) & continue to exist, become β_0-family.
→ The discs in class β_2, which have to intersect the x_2-axis (the homotopy doesn't cross it), can't survive. At $r=1$, disc of area λ in x_1-plane bubbles off. Call its class α, and denote by w the corresponding variable. In fact $\alpha = \beta_2 - \beta_1$, and $w = z_2/z_1$.

In converse direction, as $r\uparrow$ through 1, still w/ $\lambda>0$, β_0-disc persists & becomes β_1 but as r crosses 1 we can attach the α-disc of $\mu=0$ to it (note $\#(\partial\beta_1 \cap \partial\alpha) = 1$), to build a new disc in class $\beta_0 + \alpha = \beta_2$.

So... the right way to match W_L's to account for this is $z_1\underbrace{(1+w)} = z_0$

accounts for $\mu=0$ configurations ($\emptyset$ or α) that can split off/attach at wall.

If we had any other class β with $\partial\beta\cdot\partial\alpha = k$ (signed intersection number),
then would be k places at the boundary of a disc in class β where α-disc can be attached to form discs in classes $\beta + j\alpha$ $j = 0 \dots k$.

The wall-crossing transformation is then $z^{\partial\beta} \mapsto z^{\partial\beta}(1+w)^{\partial\beta\cdot\partial\alpha}$

This is indeed a coord. change on W_L. (but not clear a priori we remain among Laurent poly's)

Here the fact that $W_{prod} = z_1 + z_2 = z_1(1+w)$ remains a Laurent poly after subs. $z_1 = \frac{z_0}{1+w}$ follows from $\exists$ of monotone tori (Chekanov) in the other chamber.

[In general, in non-monotone settings one should return to a Novikov field, setting eg.
$W_L = \sum_{\mu(\beta)=2} n_\beta q^\beta$ in completion of $\mathbb{Z}[H_2(M,L)]$ wrt sympl. area (ie. ∞ sums with $\omega(\beta_i) \to +\infty$)
or more common, use weights $t^{\omega(\beta)} z^{\partial\beta}$ as before (in monotone case, all powers of t are the same & we can forget t).
$\uparrow$ formal variable, completion wrt t real exponents $\to +\infty$]

- The product $\leftrightarrow$ Chekanov wall-crossing transf. $(z_1, z_2) \sim (z_1, w = \frac{z_2}{z_1}) \mapsto (z = z_1(1+w), w)$
also relates in $\mathbb{CP}^2$ $W_{cl} = z_1 + z_2 + \frac{1}{z_1 z_2} = \underbrace{z_1 + z_1 w}_{\text{attach } \phi \text{ or } \alpha} + \frac{1}{z_1^2 w}$

$\rightsquigarrow W_{ch} = z + \frac{\overbrace{(1+w)^2}^{\text{2 places on } u(\partial D^2) \text{ to attach } \phi \text{ or } \alpha \text{ at each.}}}{z^2 w}$.

- In general case, wall-crossing transformation is a change of variables involving a generating series counting all $\mu = 0$ discs which occur - in a suitable manner
($\triangle$ multiple covers, lack of regularity of $\mu = 0$ families, ...)
eg. for Chekanov-Schlenk vs. product, it is $z = z_1(1 + w_1 + \dots + w_{n-1})$
when T intersects a coord. plane $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ in a product torus.
__All__ discs bounded by it inside $\mathbb{C}^{n-1}$ occur as $\mu = 0$, but somehow the only visible terms in the wall-crossing transformation involve the basic $D^2 \times pt \times pt$ discs, not any of the higher degree discs inside $\mathbb{C}^{n-1}$!

- General formalism + wall-crossing as change of var's follows from FOOO (ch. 4 of book)
2D formula: Pascaleff-Tonkonog (after earlier work of DA & P. Seidel).
for mutations of monotone Lagr. tori — Tonkonog-Vianna
$\downarrow$ (tropical geometry counterpart: cf. Gross-Siebert & Kontsevich-Soibelman).
using "tropical discs" instead of actually computing moduli spaces of holomorphic discs.